

Analytical improvements to the Breit-Wigner isobar models

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We discuss the derivation and properties of the general representation of partial wave amplitudes in the context of improving the models currently used in analysis of three particle Dalitz distributions.

I. INTRODUCTION

In this note, after a brief introduction to aspects of S -matrix theory relevant in analysis of three particle Dalitz plots, I focus on properties of Breit-Wigner (BW) amplitudes and the isobar model in general. I discuss the LHCb analysis model in the context of a general isobar-type approximation and show, for example, which features of the BW amplitude, *e.g.* barrier factors, Blatt-Weisskopf factors, *etc.* are universal and which are not, *i.e.* are process dependent. The possibility of extending the BW description in a way that is consistent with analyticity, unitarity, and even crossing would allow to access systematic uncertainties in data analysis. I concentrate on spinless particles. Spin introduces kinematical complexities but does not affect how unitarity, analyticity, and crossing are implemented, at least for a finite set of partial waves.

II. KINEMATICAL VS DYNAMICAL SINGULARITIES

We are interested in amplitudes describing a decay of a quasi-stable particle D with mass M to three distinguishable particles A, B, C

$$D \rightarrow A + B + C \quad (1)$$

The decay amplitude depends on particle helicities, λ_i , $i = A, B, C, D$, and three Mandelstam invariants, which we define as $s = (p_A + p_B)^2$, $t = (p_B + p_C)^2$ and $u = (p_A + p_C)^2$. The invariants are kinematically constrained by $s + t + u = \sum_i m_i^2$. Analytical S -matrix theory states that, besides the decay channel, the same amplitude describes each of the three two-to-two scattering processes, *i.e.* the s -channel reaction $D + \bar{C} \rightarrow A + B$, (bar denotes an antiparticle) as well as the t and u channel scattering. What this means in practice is the following. For each combination of helicities there is an analytical function $A_{\lambda_i}(s, t, u)$ of the three complex Mandelstam variables and complex M^2 , such that the three physical scattering amplitudes and the decay channel amplitude correspond to the limit of A_{λ_i} when s, t, u , and M approach the real axis in the physical domain of the corresponding reaction. This is the essence of crossing symmetry. In general crossing mixes helicity amplitudes and

leads to complicated relations for helicity amplitudes. Furthermore, helicity amplitudes have kinematical singularities in the Mandelstam variables. Despite such complexities it is possible to come up with parameterizations of helicity amplitudes that take into account both kinematical and dynamical constraints [1]. On the other hand it is also useful to consider the covariant form *i.e.* a representation of helicity amplitudes in terms of Lorentz-Dirac factors that describe wave functions of the free particles participating in the reaction. The advantage of the covariant representation is that the scalar functions multiplying all independent covariants are simply related by crossing and are free from kinematical singularities. At the end of the day one still needs the helicity amplitudes for partial wave analysis. In [2] the reaction $\omega \rightarrow \pi^+ \pi^- \pi^0$ was studied and this example provides a good illustration of the issues discussed above.

The main postulate of relativistic reaction theory is that reaction amplitudes are analytical functions of kinematical variables. It follows from Cauchy's theorem that an analytical function is fully determined by its domain of analyticity, *i.e.* location of singularities. Thus knowing amplitude singularities allows to determine the amplitude elsewhere, including the various physical regions. In S -matrix theory it is assumed that all singularities can be traced to unitarity. In absence of an explicit solution to the scattering problem in QCD, analyticity and unitarity provide the least model dependent description of hadron scattering.

Unitarity operates in any of the Mandelstam variables. In the s -channel the physical domain is located on the positive real axis in the s -plane above the elastic threshold. Unitarity makes amplitudes singular at each open channel threshold, the singularity being of the square-root type. The same happens in variables u and t . The amplitude also has singularities in the variable M since it represents an unstable channel. In studying a particular decay process, *e.g.* $J/\psi \rightarrow 3\pi$, M is fixed in a very narrow range (within the width of the J/ψ), and dependence on M is effectively fixed and its singularity structure irrelevant.

In general, it is not known how to write an amplitude that has correct unitarity constraints in two or more overlapping channels. The reason being that it is simple to implement unitarity on partial waves where it is an algebraic constraint. A single partial wave in one channel corresponds to an infinite number of partial waves in another one thus imposing unitarity on say both s and t

channel partial waves simultaneously requires an infinite number of partial waves. Regge theory extends the concept of analytical continuation to the angular momentum variable of partial waves. Singularities in the angular momentum plane *e.g.* Regge poles, determine behavior of infinite sums of partial waves, thus Regge theory is used to implement cross channel unitarity.

In general amplitude singularities are known only in a limited domain but as long as they dominate in a kinematical region of interest one may be able to construct a realistic amplitude model. Amplitude models fall into two main categories. One is that of dual models, *e.g.* the Veneziano model and the other is the isobar model category, *e.g.* the Khuri-Treiman model. Dual models attempt to incorporate S -matrix constraints directly on the full amplitude that depends on the Mandelstam variables. Since isobar is synonymous with a partial wave, isobar models are models based on a (truncated) partial wave expansion.

In the following I will discuss the isobar model in some detail since this is almost exclusively the model used at present in analyses of three particle Dalitz distributions.

III. THE BREIT-WIGNER AMPLITUDE

In the Breit-Wigner formula, a partial wave, $f_l(s)$ is approximated by a pole in s located in the complex energy plane at $s_P = \text{Re } s_P - i \text{Im } s_P$,

$$f_l^p(s) \propto \frac{1}{s_P - s} \quad (2)$$

The real and imaginary parts are related to the mass, $M = \sqrt{\text{Re } s_P}$ and the width, $\Gamma = \text{Im } s_P / M$ of a resonance. For comparison with experiment a reaction amplitude is evaluated at physical, real values of kinematical variables *e.g.* when s approaches the real axis from above. The contribution from a BW pole to a partial cross section, σ_l is proportional to

$$\sigma_l \propto \lim_{\epsilon \rightarrow 0} |f_l^p(s + i\epsilon)|^2 \propto \frac{1}{(s - M^2)^2 + (M\Gamma)^2}. \quad (3)$$

Since the resonance pole is located in the complex s -plane, on the real s -axis, where experimental data is taken, it produces a smooth variation in the cross section. The closer the pole is to the real axis, *i.e.* the smaller the resonances width, the more rapid is the variation in cross section or event distribution. It is worth keeping in mind that once energy is considered as a complex variable any variation of the reaction amplitude in the physical region can be traced to existence of singularities in complex energy plane *e.g.* a pole as in the BW formula.

IV. UNITARITY AND THE BREIT-WIGNER AMPLITUDE

The BW formula of Eq. (2) is an analytical function of s except for a single pole at $s = s_P$. How does unitarity constrain the BW pole? Resonance decay is possible because of open channels and it is unitarity that controls distribution of probability across decays. It thus follows that unitarity must constrain resonance decay widths and thus the imaginary part of the BW pole. But Eq. (2) is only an approximation to the “true”, unitary amplitude valid for s near the position of the complex pole. Since unitarity operates in the physical domain, *i.e.* on the real axis, the constraint of unitarity on the “true” amplitude is lost in the pole approximation, *i.e.* at a finite distance from the real axis. In this case implementing unitarity is related to using energy dependent widths.

Suppose the lowest mass open channel is a state of particles A and B with threshold at $s = s_{th} = (m_A + m_B)^2$. In the mass range between s_{th} and the first threshold, unitarity constrains the “true” amplitude to satisfy,

$$\text{Im } \hat{f}_l(s) = \hat{t}_l^*(s) \rho_l(s) \hat{f}_l(s) \quad (4)$$

Here $\hat{f}_l(s) = f_l(s)/(aq)^l$ and $\hat{t}_l(s) = t_l(s)/(aq)^{2l}$ are the s -channel reduced partial waves representing production of AB in $D + \bar{C} \rightarrow A + B$ and elastic $A + B \rightarrow A + B$ scattering, respectively. Near threshold $q \rightarrow 0$, with q being the relative momentum between A and B in the s -channel center of mass frame, partial waves vanish as $(aq)^l$, where a is given by the position of the lowest mass singularity in the crossed channels, *i.e.* the range of interaction. $\rho_l(s)$ is a known kinematical function describing the two-body phase space. It has a square-root branch point at $s = s_{tr}$. As s increases past the first inelastic threshold, $\text{Im } \hat{f}_l(s)$ receives a “kick” from another square-root type singularity from channel openings and the *r.h.s* of Eq. (4) needs to be modified. Eventually three and more particle channels open. In practice unitarity is a useful constraint in a limited energy range that covers a small number of open channels *e.g.* close to the elastic threshold. Replacing, in Eq. (4), \hat{f}_l by \hat{t} one obtains the unitarity relation for the elastic $A + B \rightarrow A + B$ partial waves in the elastic region,

$$\text{Im } \hat{t}_l(s) = |\hat{t}(s)|^2 \rho_l(s) \quad (5)$$

Above the inelastic threshold the *r.h.s* of Eq. (5) should be modified as discussed above. In case there is a finite number, N of relevant inelastic channels the unitarity condition can be expressed in a matrix form

$$\text{Im } \hat{t}_{l,ij}(s) = \sum_k \hat{t}_{l,ik}^*(s) \rho_{l,k}(s) \hat{t}_{l,kj}(s) \quad (6)$$

where $\hat{t}_{l,ij}(s) = t_l(s)/(aq)^{l_i}(aq)^{l_j}$ and $\rho_{l,k}(s)$ is the appropriate, reduced phase space in the channel k . The

unitarity relation for $\hat{f}_{l,i}(s)$ takes on a similar form,

$$Im \hat{f}_{l,i}(s) = \sum_k \hat{t}_{l,ik}^*(s) \rho_{l,k}(s) \hat{f}_k(s) \quad (7)$$

For a given $\hat{t}_l(s)$, the analytical amplitude $\hat{f}_l(s)$ that satisfies Eq. (4) can be written as

$$\hat{f}_l(s) = \hat{t}_l(s) G_l(s). \quad (8)$$

and sometimes the so-called Muskhelishvili-Omnès function is used instead of $\hat{t}_l(s)$ on the *r.h.s* of Eq. (8) [3]. The function $G_l(s)$ is an analytical function of s with cuts except on the real axis in the elastic region. The latter are accounted for by the elastic amplitude $t_l(s)$. The singularities of $G_l(s)$ correspond to the (often unknown) contributions from the unitarity-demanded left hand cuts that exist in the crossed, t and u channels. The inelastic contributions *i.e.* right cuts in \hat{f}_l are related to the inelastic contributions to the amplitude $\hat{t}_l(s)$ which is easy to show if the matrix representation is used,

$$\hat{f}_{l,i} = \sum_k \hat{t}_{l,ik} G_{l,k}(s) \quad (9)$$

and with the functions $G_{l,k}(s)$ bearing only left hand cuts. Now we go back to the pole formula. From Eq. (4) it follows that poles of the production amplitude \hat{f}_l are also poles of \hat{t}_l . This is because no resonance poles appear on sheets connected to the left hand cuts [4]. Since $\hat{t}_l(s)$ satisfies Eq. (5) it can be shown that the most general parametrization has the form,

$$t_l(s) = \frac{1}{C_l(s) - I_l(s)} \quad (10)$$

which in the inelastic case generalizes to the matrix form

$$t_{l,ij}(s) = [C_l(s) - I_l(s)]_{ij}^{-1} \quad (11)$$

with C_l and I_l becoming $N \times N$ matrices in the channel space. The function $C_l(s)$ ($C_{l,ij}(s)$) has similar properties to the function $G_l(s)$ ($G_{l,i}(s)$) in Eq. (8), *i.e.* they are real for real s in the elastic region and have only left hand cuts. The function $I_l(s)$ is a known analytical function *i.e.* the Chew-Mandelstam function, with its imaginary part for real s given by $Im I_l(s) = \rho_l(s)$. The reason why the analytical solution of Eq. (5) is more complicated than that of Eq. (4), is that the former is a non-linear relation for the amplitude. It is easy to check that this equation becomes a linear condition for the inverse of \hat{t}_l and this is the reason why dependence on phase space appears in the denominator in Eq. (10). It is straightforward to check that Eq. (10) satisfies Eq. (4), or in the inelastic case, its matrix generalization. Presence of “denominators” in amplitudes are a direct consequence of unitarity and so are the resonance poles, which correspond to zeros of the denominators.

One immediately recognizes that the K -matrix, or the K function in the elastic case corresponds to

$$K_l^{-1}(s) = C_l(s) - Re I_l(s). \quad (12)$$

Since the real part of a function is not an analytical function an analytical approximation to the K matrix violates analyticity of the amplitude and may lead to spurious “kinks” from square-root unitarity branch points in the physical region. It is much better to use Eq. (10), *aka* the Chew-Mandelstam representation, with the analytical (K -matrix type) parametrization reserved for the analytical function $C_l(s)$.

Combining Eq. (10) with Eq. (8) one obtains

$$\hat{f}_l(s) = \frac{G_l(s)}{C_l(s) - I_l(s)} \quad (13)$$

or in the matrix form for the inelastic case

$$\hat{f}_{l,i}(s) = \sum_k [C_l(s) - I_l(s)]_{i,k}^{-1} G_{l,k} \quad (14)$$

Now we can finally see how the BW pole formula of Eq. (2) emerges. Suppose in Eq. (13) the denominator vanishes at some complex $s = s_p$. Near the pole of \hat{f}_l

$$\hat{f}_l(s) \sim \frac{\beta_l}{s - s_p} \quad (15)$$

where

$$\beta_l = G_l(s_p) / (C'_l(s_p) - I'_l(s_p)) \quad (16)$$

In the inelastic case the role of the denominator in the *r.h.s* of Eq. (13) is played by the determinant of the $N \times N$ matrix $[C_l(s) - I_l(s)]$. Even though the residue of the pole, β_l is in general is a complex number, it can be shown that only its magnitude is to be related with a coupling of a resonance to a decay channel [4]. The residue β_l should be distinguished from the numerator $G_l(s)$. The latter is energy dependent and represents production amplitude of the final state $A+B$ given the initial state $D+\bar{C}$. The former is a number representing the product of couplings of the resonance to the initial and final states.

Finally we “derive” the more familiar BW formula, with energy dependent widths. In the “LHCb” notation

$$f_{l_s}^{LHCb}(s) = F_{l_s}(q) R_{l_s}(s) F_{l_t}(p) \quad (17)$$

so that the reduced amplitude is given by

$$\hat{f}_{l_s}^{LHCb}(s) = \frac{F_{l_s}(q)}{(aq)^{l_s}} R_{l_s}(s) F_{l_t}(p) \quad (18)$$

where $q = q(s)$ is the decay channel relative momentum between A and B and $p = p(s)$ is the decay channel relative momentum between the (AB) pair and the spectator particle C . For comparison with the analysis given above all is needed is to replace the decay channel expressions

for q and p by the s -channel ones. The function $F_l(x)$ is a product of an angular momentum barrier factor x^l and a Blatt-Weisskopf factor

$$F_l(x) = (ax)^l F'_l(x) \quad (19)$$

where, for example,

$$F'_2(q) = \sqrt{\frac{13}{((aq) - 3)^2 + 9(aq)}} \quad (20)$$

The propagator $R_{l_s}(s)$ is given by

$$\begin{aligned} R_{l_s}(s) &= \frac{1}{m_r^2 - s - i\rho_{l_s}(s)X(p)} \\ &= \frac{1}{X(p)(C^{LHCb}(s) - i\rho_{l_s}(s))}, \quad C^{LHCb}(s) \equiv \frac{m_r^2 - s}{X(p)} \end{aligned} \quad (21)$$

so that one can rewrite the LHCb amplitude model as

$$\hat{f}_{l_s}^{LHCb}(s) = \frac{G_{l_s}^{LHCb}(s)}{C^{LHCb}(s) - i\rho_{l_s}(s)} \quad (22)$$

where

$$G_{l_s}^{LHCb}(s) = \frac{F'_{l_s}(q)F_{l_t}(p)}{X(s)} \quad (23)$$

Eq. (22) is a specific case of Eq. (13). It is in fact the K matrix (function) approximation since only the imaginary part, $\rho_{l_s}(s)$ of the dispersive integral $I_l(s)$ is used. The functions C_l and G_l containing, through left hand cuts, physics of elastic production of AB in $A + B \rightarrow A + B$ and in decay $D \rightarrow A + B + C$, respectively, have been replaced by a specific product of Blatt-Weisskopf factors. The latter originate from a potential model in non-relativistic scattering and at best can be considered as a crude approximation. In precision data analysis they should be replaced by a more flexible parametrization. In LHCb analysis contribution from several poles are included by adding BW amplitudes. Eq. (13) shows how all such poles need to appear as zeros of the common denominator. Finally the matrix representation of Eq. (14) is the correct formula for dealing with multiple channels.

V. COMBINING s , t , AND u , CHANNEL ISOBARS AND CORRECTIONS TO THE ISOBAR MODEL

In the previous section we discussed how the energy dependent BW amplitude is related to the general expression for the partial wave. Here we discuss how the partial wave amplitudes build the full amplitude in the decay channel.

In the scattering domain of the s -channel partial wave expansion of the full amplitude converges and a finite

number of partial waves may give a good approximation to the whole sum. Energy dependence of individual partial wave can be represented using expressions like the one in Eq. (8) or Eq. (9). In a decay channel partial wave series also converges, but one cannot simply replace the one sum but the other. This is because the decay channel, partial waves have extra "complexity" compared to the scattering channel, due to t and u channel singularities, *i.e.* resonances begin in the physical region. Ignoring spins of external particles, the s -channel partial wave series is given by

$$A(s, t, u) = \frac{1}{4\pi} \sum_{l=0}^{\infty} (2l+1) f_l(s) P_l(z_s) \quad (24)$$

On the $r.h.s$ the dependence on t and u is algebraic, through z_s and the rotational functions. Thus in the physical domain of the decay the s -channel series diverges because the $l.h.s$ has singularities in t and u . Furthermore, it follows that any truncated, finite set of s -channel partial waves cannot reproduce t or u -channel singularities, *e.g.* resonance which appear inside the Dalitz plot. These issues are resolved in the isobar model by replacing the infinite number of s -channel partial waves by a finite set and adding a (finite) set of t and u channel partial waves. Thus the amplitude has a mixed form that includes partial waves (isobars) in the three channels simultaneously,

$$A(s, t, u) = \frac{1}{4\pi} \sum_{l=0}^{L_{max}} (2l+1) a_l^{(s)}(s) P_l(z_s) + (s \rightarrow t) + (s \rightarrow u) \quad (25)$$

We refer to $a_l^{(s)}(s)$ as the isobaric amplitude in the s -channel and analogously, $a_l^{(t)}(t)$ and $a_l^{(u)}(u)$ are the isobaric amplitudes in the t and u -channels, respectively. In a typical Dalitz plot analysis, the isobaric amplitudes are parametrized using the energy dependent BW amplitudes discussed in the previous sections.

Projecting the $r.h.s$ of Eq. (25) into the s -channel gives the s -channel partial waves, which we denoted by $f_l(s)$,

$$\begin{aligned} f_l(s) &= a_l^{(s)}(s) \\ &+ \frac{1}{2} \int_{-1}^1 dz_s P_l(z_s) \sum_{l'=0}^{L_{max}} (2l'+1) a_{l'}^{(t)}(t) P_{l'}(z_t) + (t \rightarrow u) \\ &\equiv a_l^{(s)}(s) + b_l^{(s)}(s). \end{aligned} \quad (26)$$

Under the integral, t and u are to be considered as function of s and z_s , the cosine of the s -channel scattering angle. Since the integral contributes to partial waves with arbitrary l , Eq. (25) defines a model for an infinite number of partial waves $f_l(s)$ and gives the result of the analytical continuation of the series in Eq. (24). Application of unitarity in the s -channel leads to a relation between the isobaric amplitudes

$$a_l^{(s)}(s) = t_l(s) \left[\frac{1}{\pi} \int_{s_{tr}} ds' \frac{\rho_l(s') b_l^{(s)}(s')}{s' - s} \right] \quad (27)$$

The amplitude $b_l^{(s)}(s)$ is the s -channel projection of the t and u channel exchanges. As a function of s it has the left hand cut but no right hand, unitary cut. The dispersive integral in Eq. (27) has the s -channel unitary cut. Thus s -channel unitarity demands that s -channel isobaric amplitude $a_l^{(s)}(s)$ has the right cut coming not only from the elastic $A + B \rightarrow A + B$ amplitude, $t_l(s)$, but also from dispersion of s -channel projections of the t and u -channel amplitudes. It is important to note that difference between the partial wave amplitudes $f_l(s)$ and the isobaric amplitudes $a_l^{(s)}(s)$. In case of the former the right hand cut discontinuity comes entirely from the elastic scattering, *c.f.* Eq. (8), (9). In the isobar model, the partial wave amplitudes are given by,

$$f_l(s) = t_l(s) \left[\frac{b_l^{(s)}(s)}{t_l(s)} + \frac{1}{\pi} \int_{s_{tr}} ds' \frac{\rho(s') b_l^{(s)}(s')}{s' - s} \right] \\ \equiv t_l(s) G_l(s) \quad (28)$$

which are indeed of the form given by Eq. (8), since it can be shown that the right hand cuts cancel between the two terms in the square bracket so that $G_l(s)$ has only left hand cuts,

$$G_l(s) = \frac{1}{\pi} \int_{s_{tr}} ds' \rho(s') \frac{b_l^{(s)}(s') - b_l^{(s)}(s)}{s' - s} \quad (29)$$

In the inelastic case it generalizes to the form given by Eq. (9). The left hand cuts, as expected, originate from exchanges in the crossed channels determined by the amplitude $b_l^{(s)}(s)$.

One goes further and also impose t and u -channel unitarity, thereby correlating isobar expansions in the three channels. This is the analytical, unitary description of “final state interaction” that relies on the model independent features of the amplitudes only.

The implications for event distributions in the Dalitz plot, given the left hand cut singularities of $b_l^{(s)}(s)$ were studied in [6] and in the next section we take a look at the left hand cuts of the amplitude $G_l(s)$.

VI. WATSON'S THEOREM

Since the imaginary part of a complex function is itself a real function, it follows from Eq. (4) that in the elastic region, phase of $\hat{f}_l(s)$ equals that of the elastic scattering amplitude $t_l(s)$. It is often forgotten to be mentioned, however, that in many cases the *r.h.s* in Eq. (4) does not saturate the imaginary part even in the elastic regime. When this happens, Watson's theorem is violated even in the elastic regime. And this is the case of a decay process. In this case there is nevertheless a relation between \hat{f}_l and \hat{t}_l , or a generalized Watson relation.

In the decay kinematics, even if the energy of the AB state is below inelastic threshold, the imaginary part of the s -channel partial amplitude, $\hat{f}_l(s)$ is not in given by Eq. (4). This happens because in the decay kinematics cross channel singularities (from thresholds/unitarity) *e.g.* isobar exchanges in t or u channel are located in the physical region of the s -channel and contribute to the imaginary part of the s -channel partial wave. Another way of saying this, is that in the decay kinematics singularities of $\hat{f}_l(s)$ which otherwise are to the left of the elastic unitarity cut move to the right of the elastic unitary branch point.

In the previous section we argued that Eq. (8) follows from the assumption that $\hat{f}_l(s)$ is an analytical function. The relation between analyticity and casualty applies to full amplitudes, while partial waves are related to full amplitudes in a complicated way. Analytical partial waves are obtained by a continuation of the unitary relation to the complex energy plane. It can be shown (see *e.g.* [4]) that when left and right hand cuts are separated $Im \hat{f}(s) = \Delta \hat{f}(s_+) \equiv (\hat{f}_l(s_+) - \hat{f}_l(s_-))/2i$, where $s_{\pm} = s \pm i\epsilon$. That is, the imaginary part of the amplitude as measured in the experiment, which by itself is a real function, is equal to the discontinuity across the real axis of the unique extension of $\hat{f}_l(s)$ to the complex energy plane. It turns out, however that in decay kinematics, the proper extension of $\hat{f}_l(s)$ to the complex energy plane is such that in Eq. (4) the *l.h.s* should be replaced by $\Delta \hat{f}(s_+)$ and the *r.h.s* should be replaced by the product $t_l(s_-) \rho_l(s) f_l(s_+)$. Thus, elastic unitarity still determines the discontinuity across the right hand cut but it is no longer a real function. This is the generalized Watson or discontinuity relation.

As a result the representation given by Eqs. (8),(9) is still valid with the exception that the numerators, $G_l(s)$ become complex in the elastic region. They do not have the unitarity cut though. Just like in the “standard” Watson's theorem the right hand cut discontinuity comes entirely from the elastic amplitude $t_l(s)$. The complexity of $G_l(s)$ is still of the left hand cut nature, except that left hand cuts have moved onto the right hand side under the right hand cut and into the second sheet. [5].

ACKNOWLEDGMENTS

I would like to thank the organizers and participants of the LHCb workshop on multi-body decays of B and D mesons for stimulating discussions. This work is supported by the U.S. Department of Energy, Office of Science, Office of Nuclear Physics under contract DE-AC05-06OR23177. It is also supported in part by the U.S. Department of Energy under Grant No. DE-FG0287ER40365.

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